Counting Yang-Mills Dyons with Index Theorems

Mark Stern[*,1] and Piljin Yi[+,2]

[*] *Department of Mathematics, Duke University, Durham, NC 27706, USA*

[+] *School of Physics, Korea Institute for Advanced Study*
207-43, Cheongryangri-Dong, Dongdaemun-Gu, Seoul 130-012, Korea

ABSTRACT

We count the supersymmetric bound states of many distinct BPS monopoles in $N = 4$ Yang-Mills theories and in pure $N = 2$ Yang-Mills theories. The novelty here is that we work in generic Coulombic vacua where more than one adjoint Higgs fields are turned on. The number of purely magnetic bound states is again found to be consistent with the electromagnetic duality of the $N = 4$ $SU(n)$ theory, as expected. We also count dyons of generic electric charges, which correspond to 1/4 BPS dyons in $N = 4$ theories and 1/2 BPS dyons in $N = 2$ theories. Surprisingly, the degeneracy of dyons is typically much larger than would be accounted for by a single supermultiplet of appropriate angular momentum, implying many supermutiplets of the same charge and the same mass.

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[1] E-mail: stern@math.duke.edu,

[2] E-mail: piljin@kias.re.kr

1 Introduction

The purpose of this paper is to count BPS monopoles and dyons in supersymmetric Yang-Mills theories. Such computations have been performed by many authors using moduli space dynamics of monopoles [1, 2, 3, 4, 5]. Recently [6, 7, 8], however, it was realized that the monopole dynamics in a generic vacuum is qualitatively different from the old moduli space dynamics of Manton [9] employed in most such endeavors, where the low energy dynamics of monopoles were considered when only one adjoint Higgs field is turned on, while supersymmetric Yang-Mills field theories come with 2 or 6 such scalars. This restriction disallows static interaction between monopoles [10], so that all interaction comes from nontrivial coefficients of kinetic terms. In a generic vacuum with more than one adjoint Higgs turned on, monopoles of the same type still have no static force among them, but dynamics of monopoles of distinct type could have a static potential. In this article, we solve various index problems as a first step towards counting all BPS states.

In the old monopole dynamics, the Lagrangian one finds is a pure sigma model with the moduli space as the target manifold. Classically, one solves for geodesic trajectories to find classical orbits of monopoles. Quantum mechanically, the Hamiltonian is a square of a supercharge which can be regarded as a Dirac operator acting either on the spinor bundle or on the Clifford bundle over the moduli space,

$$Q \sim -i\gamma^m \nabla_m. \tag{1}$$

Supersymmetric bound states, for example, would be found as normalizable spinors or forms on the moduli space that are also zero modes of this Dirac operator.

The new supersymmetric low energy dynamics is obtained by augmenting old moduli space dynamics with a set of supersymmetric potential terms, and was written explicitly in Ref.[7, 8]. In principle, the question of BPS states must be reconsidered in the new dynamics. (One example of states that cannot be probed in the old formalism is the now well-known 1/4 BPS states [11, 12, 13].) In this new setting, the supercharges of the low energy dynamics will again be interpreted as Dirac operators on the moduli space, which is now twisted by some triholomorphic Killing vector field, say K,

$$Q \sim \gamma^m(-i\nabla_m - K_m). \tag{2}$$

The corresponding bosonic potential is precisely half the squared norm of K [14, 6, 7]. We count the number of normalizable states annihilated by a Dirac operator, weighted

by ± 1 for the chiral and the antichiral states respectively. The resulting integer is the index of the Dirac operator. In fact this index would be infinite or ill defined without further restriction of the domain of the Dirac operator. We restrict the problem to each charge eigensector before counting the zero modes of the Dirac operator. Thus, effectively, we will be computing an equivariant index with L^2 condition. This gives information on the existence and the degeneracy of dyonic bound states for each electric charge sector.

Understanding monopole dynamics in generic vacua is particularly significant in the context of $N = 2$ Yang-Mills theories, because many (1/2) BPS dyons exist only when both adjoint Higgs are turned on. By ignoring the potential term in the low energy dynamics, one would in effect be searching for a bound state in the vacua where the state cannot exist as a supersymmetric one-particle state. In the language of Seiberg-Witten [15], one would be looking for it on one side of a marginal stability domain wall in the vacuum moduli space, while the bound state in question exists only on the other side of the domain wall. For example, most dyons that become massless at special hypersurfaces in the Seiberg-Witten moduli space, are of this type [16, 17, 18] and cannot be probed by old moduli space dynamics.

In Section 2, we review the supersymmetric quantum mechanics with potential for the case of 4 and 8 supercharges. We isolate various involutions, with respect to which the index is defined. Section 3 introduces the explicit form of the moduli space metric that governs the dynamics of BPS monopoles. The only known moduli space for arbitrarily many monopoles is the case of all distinct monopoles, and this is the case for which we will compute the index explicitly. Section 4 recalls a recent explicit computation of two-monopole bound states in $SU(3)$ theories and presents the resulting value of the indices. In section 5, we finally delve into the computation of the index by using a Fredholm deformation of the Dirac operator in question. The computation reduces to that of a superharmonic oscillator in 4 dimensions, whose index is computed explicitly. Section 6 translates the results to degeneracies of various dyonic and purely magnetic bound states and checks its consistency with anticipated nonperturbative physics. We close with a summary.

2   Supersymmetric Sigma Model with Potential

In this section, we briefly review the supersymmetric sigma-model quantum mechanics with potentials. These quantum mechanics have been introduced as the low energy

dynamics of monopoles in pure $N = 2$ Yang-Mills field theory and in $N = 4$ Yang-Mills field theory [7, 8]. They also appeared in other systems such as the dynamics of instanton solitons [19].

2.1 Quantum Mechanics with 4 Real SUSY

The SUSY dynamics we consider is a sigma model with potential, whose Lagrangian is written compactly as

$$\mathcal{L} = \frac{1}{2}\left(g_{mn}\dot{z}^m\dot{z}^n + ig_{mn}\lambda^m D_t\lambda^n - g^{mn}G_mG_n - i\nabla_m G_n\lambda^m\lambda^n\right), \tag{3}$$

where $D_t\lambda^m = \dot{\lambda}^m + \Gamma^m_{np}\dot{z}^n\lambda^p$. The target manifold must be hyperKähler, which means that there are three covariantly constant complex structures $J^{(s)}$ satisfying quaternionic algebra,

$$J^{(s)}J^{(t)} = -\delta^{st} + \epsilon^{stu}J^{(u)}, \tag{4}$$

and the Killing vector field G should be triholomorphic.

$$\mathcal{L}_G g = 0, \qquad \mathcal{L}_G J^{(s)} = 0. \tag{5}$$

Introducing vielbein e_m^E and defining $\lambda^E = \lambda^m e_m^E$ which commute with all bosonic variables, the canonical commutators are

$$
\begin{aligned}
[z^m, p_n] &= i\delta^m_n, \\
\{\lambda^E, \lambda^F\} &= \delta^{EF}.
\end{aligned} \tag{6}
$$

We can realize this algebra on spinors on the moduli space by letting $\lambda^E = \gamma^E/\sqrt{2}$, where γ^E are gamma matrices. (Since the moduli space is hyperKähler an equivalent quantization is obtained using holomorphic differential forms.) The supercovariant momentum operator, defined by

$$\pi_m = p_m - \frac{i}{4}\omega_{mEF}[\lambda^E, \lambda^F], \tag{7}$$

where $\omega_{m\,E}^{\ \ F}$ is the spin connection, then becomes the covariant derivative acting on spinors $\pi_m = -i\nabla_m$. Note that

$$
\begin{aligned}
[\pi_m, \lambda^n] &= i\Gamma^n_{mp}\lambda^p, \\
[\pi_m, \pi_n] &= -\frac{1}{2}R_{mnpq}\lambda^p\lambda^q.
\end{aligned} \tag{8}
$$

3

The supersymmetry charges take the form

$$
\begin{aligned}
Q &= \lambda^m(\pi_m - G_m), \\
Q^{(s)} &= \lambda^m J^{(s)n}_m(\pi_n - G_n),
\end{aligned}
\tag{9}
$$

which obey

$$
\begin{aligned}
\{Q, Q\} &= 2(\mathcal{H} - \mathcal{Z}), \\
\{Q^{(s)}, Q^{(t)}\} &= 2\,\delta_{st}(\mathcal{H} - \mathcal{Z}), \\
\{Q, Q^{(s)}\} &= 0.
\end{aligned}
\tag{10}
$$

Here the Hamiltonian $\mathcal{H}$ and the central charge $\mathcal{Z}$ are given by

$$
\mathcal{H} = \frac{1}{2}\left(\frac{1}{\sqrt{g}}\pi_m \sqrt{g}\,g^{mn}\pi_n + G_m G^m + i\nabla_m G_n \lambda^m \lambda^n\right),
\tag{11}
$$

$$
\mathcal{Z} = G^m \pi_m - \frac{i}{2}(\nabla_m G_n)\lambda^m \lambda^n.
\tag{12}
$$

Note that the operator $i\mathcal{Z}$ is the Lie derivative $\mathcal{L}_G$ acting on spinors (see *e.g.*, [20])

$$
\mathcal{L}_G \equiv G^m \nabla_m + \frac{1}{8}\nabla_m G_n[\gamma^m, \gamma^n].
\tag{13}
$$

The SUSY quantum mechanics comes with a natural Z_2 grading defined by the operator,

$$
\tau_2 = \prod 2^{1/2}\lambda^E = \prod \gamma^E,
\tag{14}
$$

which anticommutes with the Dirac operator,

$$
D = \sqrt{2}\,Q = \gamma^m(-i\nabla_m - G_m).
\tag{15}
$$

This pair defines the Witten index that counts the difference between the number of bosonic states and the number of fermionic states annihilated by the supercharge. In fact, the index is defined in each superselection sector with fixed $\mathcal{Z}$, and effectively counts the difference in the numbers of BPS states of given central charges. The index will be denoted collectively by $\mathcal{I}_2$. See Section 5 for detailed computation of $\mathcal{I}_2$.

2.2   Quantum Mechanics with 4 Complex SUSY

When the number of supercharges and the number of fermions double, we obtain the following form of sigma model with potential,

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\Big(g_{mn}\dot{z}^m \dot{z}^n + ig_{mn}\bar{\psi}^m \gamma^0 D_t \psi^n + \frac{1}{6}R_{mnpq}\bar{\psi}^m \psi^p \bar{\psi}^n \psi^q \\
&\quad - g_{mn}G_I^m G_I^n - i\nabla_m G_{In}\,\bar{\psi}^m(\Omega^I \psi)^n\Big),
\end{aligned}
\tag{16}
$$

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where ψ^m is a two component Majorana spinor, $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_1, \gamma^2 = -i\sigma_3$, $\bar{\psi} = \psi^T \gamma^0$. The operator Ω_I's are defined respectively by $\Omega_4 = \delta_n^m \gamma_{\alpha\beta}^1$, $\Omega_5 = \delta_n^m \gamma_{\alpha\beta}^2$ and $\Omega_s = iJ^{(s)m}{}_n \delta_{\alpha\beta}$ for $s = 1, 2, 3$. The supersymmetry algebra again requires the manifold to be hyperKähler. As in the previous subsection, the G^I's must be triholomorphic Killing vector fields.

When quantized, the spinors $\psi^E = e_m^E \psi^m$ with vielbein e_m^E, commute with all the bosonic dynamical variables, especially with p's that are canonical momenta of the coordinate z's. The remaining fundamental commutation relations are

$$[z^m, p_n] = i\delta_n^m,$$
$$\{\psi_\alpha^E, \psi_\beta^F\} = \delta^{EF} \delta_{\alpha\beta}. \tag{17}$$

Define supercovariant momenta by

$$\pi_m \equiv p_m - \frac{i}{2}\omega_{EFm}\bar{\psi}^E \gamma^0 \psi^F, \tag{18}$$

where ω_{EFm} is the spin connection. The N=4 SUSY generators in real spinors can be written as,

$$Q_\alpha = \psi_\alpha^m \pi_m - (\gamma^0 \Omega^I \psi)^m G_m^I, \tag{19}$$

$$Q_\alpha^{(s)} = (J^{(s)}\psi)_\alpha^m \pi_m - (\gamma^0 J^{(s)}\Omega^I \psi)^m G_m^I. \tag{20}$$

These charges satisfy the $N = 4$ complex superalgebra:

$$\{Q_\alpha, Q_\beta\} = \{Q_\alpha^{(s)}, Q_\beta^{(s)}\} = 2\delta_{\alpha\beta} \mathcal{H} - 2(\gamma^0\gamma^1)_{\alpha\beta} \mathcal{Z}_4 - 2(\gamma^0\gamma^2)_{\alpha\beta} \mathcal{Z}_5, \tag{21}$$

$$\{Q_\alpha, Q_\beta^{(s)}\} = 2\gamma_{\alpha\beta}^0 \mathcal{Z}_s, \quad \{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = 2\gamma_{\alpha\beta}^0 \mathcal{Z}_3, \tag{22}$$

$$\{Q_\alpha^{(2)}, Q_\beta^{(3)}\} = 2\gamma_{\alpha\beta}^0 \mathcal{Z}_1, \quad \{Q_\alpha^{(3)}, Q_\beta^{(1)}\} = 2\gamma_{\alpha\beta}^0 \mathcal{Z}_2, \tag{23}$$

where $\mathcal{H}$ is the Hamiltonian, and the $\mathcal{Z}_I$'s are central charges,

$$\mathcal{Z}_I = G_I^m \pi_m - \frac{i}{2}\nabla_m G_n^I \bar{\psi}^m \gamma^0 \psi^n. \tag{24}$$

The sigma-model without the potential possesses an $SO(5)$ R-symmetry which is explicitly broken by the G^I's. The G^I's transform as $\mathbf{5}$ of $SO(5)_R$.

The complex form of the supercharges is often useful. To this end, we introduce $\varphi^m \equiv \frac{1}{\sqrt{2}}(\psi_1^m - i\psi_2^m)$ and define $Q \equiv \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$. The supercharges in (19) can be rewritten as

$$Q = \varphi^m \pi_m - \varphi^{*m}(G_m^4 - iG_m^5) - i\sum_{s=1}^3 G_m^s (J^{(s)}\varphi)^m, \tag{25}$$

$$Q^\dagger = \varphi^{*m} \pi_m - \varphi^m(G_m^4 + iG_m^5) + i\sum_{s=1}^3 G_m^s (J^{(s)}\varphi^*)^m. \tag{26}$$

The charges $Q^{(s)}$ and $Q^{(s)\dagger}$ are analogously defined from (20). The positive definite nature of the Hamiltonian can be seen easily in the anticommutator

$$\{Q, Q^\dagger\} = \{Q^{(s)}, Q^{(s)\dagger}\} = 2\mathcal{H}, \tag{27}$$

while the central charges appear in other parts of the superalgebra. For instance, we have

$$\begin{aligned}
\{Q, Q\} &= -\mathcal{Z}_4 + i\mathcal{Z}_5, \\
\{Q^\dagger, Q^\dagger\} &= -\mathcal{Z}_4 - i\mathcal{Z}_5.
\end{aligned} \tag{28}$$

Once we adopt this complex notation, it is natural to introduce an equivalent geometrical notation. Defining the vacuum state $|0\rangle$ to be annihilated by φ^{*m}'s, and using the 1-1 correspondence,

$$(\varphi^{m_1}\varphi^{m_2}\cdots\varphi^{m_k})|0\rangle \quad \leftrightarrow \quad dz^{m_1} \wedge dz^{m_2} \wedge \cdots \wedge dz^{m_k}, \tag{29}$$

we can reinterpret φ^m as the exterior product with dz^m, and $\varphi_m^* = g_{mn}\varphi^{*n}$ as the contraction with $\partial/\partial z^m$. The supercharge operators can be rewritten as,

$$\begin{aligned}
Q &= -id - \iota_{G^4-iG^5} + i\iota^\dagger_{J^{(1)}(G^1)} + i\iota^\dagger_{J^{(2)}(G^2)} + i\iota^\dagger_{J^{(3)}(G^3)}, \\
Q^\dagger &= id^\dagger - \iota^\dagger_{G^4+iG^5} - i\iota_{J^{(1)}(G^1)} - i\iota_{J^{(2)}(G^2)} - i\iota_{J^{(3)}(G^3)},
\end{aligned} \tag{30}$$

where ι_K is the contraction with the vector field K, and its conjugate $\iota_K^\dagger$ is the exterior product by the 1-form obtained from K by lowering its indices.

The SUSY quantum mechanics admit a canonical Z_2 grading, which in the geometrical notation of (29) is defined on k-forms by

$$\tau_4 \equiv (-1)^k. \tag{31}$$

or equivalently by

$$\tau_4 \equiv \prod 2\psi_1^E \psi_2^E = \prod(\varphi^{*E}\varphi^E - \varphi^E\varphi^{*E}). \tag{32}$$

The involution τ_4 anticommutes with all supercharges and determines the usual Witten index, $\mathcal{I}_4$.

In some special limits, however, there could be an additional Z_2 grading. Suppose that we have only one nonzero G^I, say G^5. The operators

$$\tau_\pm \equiv \prod(\sqrt{i}\,\varphi^E \pm \sqrt{-i}\,\varphi^{*E}). \tag{33}$$

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anticommutes with the Dirac operators defined as,

$$D_\pm \equiv iQ \pm Q^\dagger = (i\varphi^m \pm \varphi^{*m})(\pi_m \mp G_m^5) = d - \iota_{G^5} \pm i(d^\dagger - \iota_{G^5}^\dagger), \qquad (34)$$

the square of which is

$$D_\pm^2 = \pm 2i(\mathcal{H} \mp \mathcal{Z}_5). \qquad (35)$$

So the Z_2 gradings define an analog of the signature index for each choice of sign and for each charge-eigensector. A given state with nonzero $\mathcal{Z}_5$ can be annihilated by one of $D_\pm$ at most, and in fact must break at least half of the supercharges. The corresponding indices will be denoted by $\mathcal{I}_s^\pm$.

In Section 5, we will compute both $\mathcal{I}_4$ and $\mathcal{I}_s^\pm$ in such a special limit with only one of five G^I's present, which we can take to be G^5 without loss of generality. For $\mathcal{I}_4$, we may take any one of $D_\pm$ as the Dirac operator, since τ_4 anticommutes with both. A standard index theorem will then allow us to deduce $\mathcal{I}_4$ in more general setting.

3 Moduli Spaces

Moduli space dynamics of monopoles decompose into the interacting relative part and the non-interacting "center of mass" part. The latter corresponds to a 4-dimensional flat metric of the form,

$$g_{cm} = A\, d\vec{X}^2 + B\, d\xi_T^2, \qquad (36)$$

where $\vec{X}$ is a three-vector. Since we are interested in establishing existence of bound states, this part of the dynamics will be ignored for the most part.

The free center-of-mass sector generates two kinds of quantum numbers, nevertheless. One is the overall, conserved $U(1)$ charge, and the other is a supermultiplet structure generated by the fermionic partners of $\vec{X}$ and ξ_T. The resulting degeneracies, 4 and 16 for $N = 4$ real and complex supersymmetric quantum mechanics respectively, correspond to the smallest possible BPS multiplet of the underlying SUSY Yang-Mills field theories with 8 and 16 supercharges, respectively.

3.1 Distinct Monopoles

A simple case of this dynamics involves a collection of distinct monopoles in $SU(n)$ gauge theories. The interacting part of the moduli space metric is a simple generalization of four-dimensional Taub-NUT metric [21]. Without loss of generality, consider a collection of $k + 1$ distinct monopoles, whose magnetic charges are given by an

irreducible (sub)set of simple roots, $\beta_1, \ldots, \beta_{k+1}$. The simple roots satisfy relations $\beta_a^2 = 1$, $\beta_a \cdot \beta_{a+1} = -1/2$, and $\beta_a \cdot \beta_{a+b} = 0$ for $b > 1$. The relative part of the corresponding metric is

$$g = C_{ab}\, d\vec{r}_a \cdot d\vec{r}_b + \frac{4\pi^2}{e^4}(C^{-1})_{ab}(d\psi_a + \cos\theta_a d\phi_a)(d\psi_b + \cos\theta_b d\phi_b), \qquad (37)$$

where the matrix C for the relative moduli space is[1]

$$C_{ab} = \mu_{ab} + \frac{2\pi}{e^2}\delta_{ab}\frac{1}{r_a}. \qquad (38)$$

The 3-vector $\vec{r}_a$ is the relative position between the ath and $(a+1)^{th}$ monopoles,

$$\vec{r}_a = \vec{x}_{a+1} - \vec{x}_a, \qquad (39)$$

while the angles ψ_a of period 4π are related to the $U(1)$ phases of each monopole, ξ_a's (of period 2π), by

$$2\frac{\partial}{\partial\psi_a} = \frac{\partial}{\partial\xi_{a+1}} - \frac{\partial}{\partial\xi_a}$$
$$\left(\sum_{a=1}^{k+1} m_a\right)\frac{\partial}{\partial\xi_T} = \sum_{a=1}^{k+1} m_a \frac{\partial}{\partial\xi_a}. \qquad (40)$$

where ξ_T is a coordinate that appears in free center-of-mass part of the dynamics and m_a is the mass of the ath monopole.

For a generic reduced mass matrix μ, the triholomorphic Killing vector fields of this geometry are exhausted by

$$K_a = \frac{\partial}{\partial\psi_a}, \qquad (41)$$

so the vector fields G and G^I are linear combinations of K_a's with constant coefficients;

$$G = e\sum_c a_c K_c,$$
$$G^I = e\sum_c a_c^I K_c. \qquad (42)$$

The electric charges are measured by the charge operators,

$$-i\mathcal{L}_{K_a}, \qquad (43)$$

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[1]The coupling constant e will be assumed to be positive without loss of generality.

whose (half-)integer eigenvalues will be denoted by q_a. In terms of the simple roots β_a, the electric charge of a dyonic state with charge q_a's is

$$
\begin{aligned}
&e(+q_1 + q_2 + q_3 + \cdots + q_k + n/2)\beta_1 + \\
&e(-q_1 + q_2 + q_3 + \cdots + q_k + n/2)\beta_2 + \\
&e(-q_1 - q_2 + q_3 + \cdots + q_k + n/2)\beta_3 + \\
&e(-q_1 - q_2 - q_3 + \cdots + q_k + n/2)\beta_4 + \\
&\vdots \\
&e(-q_1 - q_2 - q_3 - \cdots - q_k + n/2)\beta_{k+1}.
\end{aligned}
\tag{44}
$$

where the integer n comes from quantization of an overall $U(1)$ angle and should be even or odd when $2\sum_a q_a$ is even or odd, respectively.

3.2   Unit Noncommutative Instanton

A simple deformation of the above moduli space appeared in another context recently, where one considers low energy dynamics of an instanton soliton in the 5-dimensional noncommutative $U(k+1)$ Yang-Mills theory [22]. This happens because an instanton in $S^1 \times R^3$ can be regarded as a collection of $k+1$ distinct monopoles of the underlying Yang-Mills theory [23]. When we compactify the theory on a circle of radius R, the nontrivial part of the moduli space of a single instanton soliton is given by the metric [19]

$$
g = \frac{4\pi^2 R}{\tilde{e}^2}\left(\tilde{C}_{ab}\, d\vec{r}_a \cdot d\vec{r}_b + (\tilde{C}^{-1})_{ab}(d\psi_a + \cos\theta_a d\phi_a)(d\psi_b + \cos\theta_b d\phi_b)\right),
\tag{45}
$$

where $\tilde{e}$ is the 5-dimensional Yang-Mills coupling. The matrix $\tilde{C}$ for the relative moduli space is

$$
\tilde{C}_{ab} = \nu_{ab} + \delta_{ab}\frac{1}{r_a} + \frac{1}{\,|\sum \vec{r}_a - 2\pi\vec{\zeta}/R|\,},
\tag{46}
$$

where $\vec{\zeta}$ encodes the noncommutativity. The matrix ν is determined by the Wilson line along S^1 that breaks the gauge symmetry to $U(1)^n$. When the supersymmetry of the underlying field theory is maximal with 16 supercharges, the low energy dynamics of the instanton is given by our SUSY quantum mechanics with 4 complex supercharges. When the field theory comes with 8 supercharges, the instanton dynamics is described by the SUSY quantum mechanics with 4 real supercharges.

4   Bound States of Two Distinct Monopoles

For a pair of two distinct and interacting monopoles, the dynamics have been solved for supersymmetric ground states in each charge eigensector. The geometry is that

of a Taub-NUT manifold which comes with a single triholomorphic Killing vector field K_1. Accordingly, there is only one conserved $U(1)$ charge, q_1, which labels superselection sectors.

In the pure $N = 2$ Yang-Mills case, define $\tilde{a}_1 \equiv 4\pi^2 a_1/e^3\mu$ where μ is the reduced mass and a_1 is defined by $G = e\,a_1 K_1$. The normalizable wavefunctions had been constructed by Pope in another context [24], and the number of dyonic bound states of charge q_1 was found to be [8]

$$2|q_1|, \tag{47}$$

if $0 < q_1 < \tilde{a}$ or $\tilde{a}_1 < q_1 < 0$, and

$$0, \tag{48}$$

otherwise. For each q_1, the solutions belong to the same chirality spinors, and thus contribute to the Witten index equally. Thus the Witten index $\mathcal{I}_2$ in each charge eigensectors are

$$\mathcal{I}_2 = \left\{ \begin{array}{ll} 2|q_1| & 0 < |q_1| < |\tilde{a}_1| \quad \text{and} \quad 0 < q\tilde{a}_1 \\ \\ 0 & \text{otherwise} \end{array} \right\}. \tag{49}$$

For a pair of distinct monopoles in $N = 4$ Yang-Mills [25], the five G^I's must be proportional to the single triholomorphic vector field $K_1 = \partial/\partial\psi_1$. We may rotate them into a single triholomorphic vector field, say $G^{I=5} = e\,a_1 K_1$, upon which we can define $\tilde{a}_1$ similarly as above, $\tilde{a}_1 \equiv 4\pi^2 a_1/e^3\mu$. The degeneracy is found to be

$$1, \tag{50}$$

for purely magnetic state ($q_1 = 0$), while for dyons

$$8|q_1|, \tag{51}$$

when $0 < |q_1| < |\tilde{a}_1|$, and zero otherwise. All solutions are self-dual differential forms, when we take the convention that the curvature tensor of the moduli space is self-dual.

When the central charge $\mathcal{Z}_5 = ea_1 q_1$ of the state is positive (negative), the bound state is annihilated by D_+ (D_-) only, while for $\mathcal{Z}_5 = 0$, the state is annihilated by both. For given q_1, we find

$$\mathcal{I}_s^+ = \left\{ \begin{array}{ll} 1 & q_1 = 0 \\ 8|q_1| & 0 < |q_1| < |\tilde{a}_1| \quad \text{and} \quad q_1 a_1 > 0 \\ 0 & \text{otherwise} \end{array} \right\}, \tag{52}$$

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and

$$\mathcal{I}_s^- = \begin{cases} 1 & q_1 = 0 \\ 8|q_1| & 0 < |q_1| < |\tilde{a}_1| \quad \text{and} \quad q_1 a_1 < 0 \\ 0 & \text{otherwise} \end{cases}. \tag{53}$$

The Witten index $\mathcal{I}_4$ counts the number of even forms minus the number of the odd forms. Of solutions with $q_1 \neq 0$, half are even and the other half are odd, so we find that

$$\mathcal{I}_4 = \begin{cases} 1 & q_1 = 0 \\ 0 & q_1 \neq 0 \end{cases} \tag{54}$$

regardless of a_1.

5   Index Computation

We would like to put a lower bound on the number of bound states in the above SUSY dynamics by computing indices. The index problems can be quite involved, given that the quantum mechanics involve many degrees of freedom with complicated interaction terms. However, the problem can be simplified by utilizing the invariance of the index under certain deformations. In this section we will use the invariance of the index under Fredholm deformation to simplify our index computations. Before proceeding with the computation, however, we need to restrict to the regime where a massgap exists.

5.1   Massgap

When restricted to specific charge eigensectors, the operators above may exhibit two drastically different behavior. For small charges, the sector has a massgap; the continuum part of the spectrum is bounded below by a positive gap. For large charges, the massgap disappears. This is the reason why there is an upper bound on the electric charge q_1 of bound states of two monopoles. In the two-body problems, the condition for the massgap to exist in a sector with electric charge q_1 is

$$|q_1| < \frac{4\pi^2}{e^3} \frac{|a_1|}{\mu}, \tag{55}$$

where the bosonic potential is generated by a single triholomorphic vector field $G = e\, a_1 K_1$. When we consider many distinct monopoles, the condition for the massgap to exist is equally simple:

$$|q_c| < |\tilde{a}_c|, \tag{56}$$

where

$$\tilde{a}_c \equiv \frac{4\pi^2}{e^3} \sum_{b=1}^{k} (\mu^{-1})_{cb}\, a_b, \tag{57}$$

with $G = ea_cK_c$. In the quantum mechanics with four complex supersymmetries, the same holds true provided that only one G^I, say, $G^5 = ea_cK_c$ is turned on. We will compute the indices, $\mathcal{I}_2$, $\mathcal{I}_4$, $\mathcal{I}_s^{\pm}$ assuming that all of these conditions hold.[2]

5.2   Index Generalities

First we recall basic definitions.

Definition 5.1 *A bounded linear operator $L : E_1 \to E_2$ between two Hilbert spaces is* Fredholm *if there exists a bounded operator $P : E_2 \to E_1$ such that $PL - I_1$, and $LP - I_2$ are compact operators. Here I_j denotes the identity map on E_j.*

The operator P in the above definition is called a parametrix. We will be interested in the case where L is a Dirac operator. In this case, although Dirac operators are unbounded on L_2, we may trivially make L bounded by taking E_1 to be the closure of the domain of L with respect to the norm (graph norm)

$$\|f\|_{graph}^2 \equiv \|f\|^2 + \|Lf\|^2,$$

where unsubscripted norms denote L_2 norms.

If L is a Dirac operator on a compact manifold, then it is well known to be Fredholm. In the compact case, one takes, for example, P to be the Green's operator, $\mathcal{G}$ defined to be the unique operator satisfying:

(i) $\mathcal{G}$ annihilates the kernel of $L^{\dagger}$.

(ii) The range of $\mathcal{G}$ is orthogonal to the kernel of L.

(iii) $L\mathcal{G}f = f$ for $f \in$ the image of L.

Then $\mathcal{G}$ is bounded by $1 + \lambda_1^{-1/2}$, where λ_1 is the first nonzero eigenvalue of $L^{\dagger}L$. Also, $\mathcal{G}L = I_1 - \Pi_1$, and $L\mathcal{G} = I_2 - \Pi_2$, where Π_1 and Π_2 denote the orthogonal projections onto the kernels of L and $L^{\dagger}$ respectively (and are finite rank and thus compact operators). Hence $P = \mathcal{G}$ satisfies all the conditions of the definition.

---

[2]If five $G^I = e\sum_c a_c^I K_c$'s are involved, the massgap condition generalizes to

$$(q_c)^2 < \sum_{I=1}^{5} (\tilde{a}_c^I)^2, \tag{58}$$

where $\tilde{a}_c^I$ are defined similarly as above for each G^I. However, the Indices $\mathcal{I}_s^{\pm}$ are not well-defined unless all G^I's are proportional to each other. We will discuss such generic cases in Section 6.

In the case of a Dirac operator on a noncompact manifold the preceding construction of a Greens operator may fail to yield Fredholmness for several reasons. The kernel of L or $L^\dagger$ may fail to be finite dimensional, making one of the projections not a compact operator. Also, if there is no gap in the spectrum, $\mathcal{G}$ will fail to be bounded. These deficits are all avoided, however, under the assumption that the essential spectrum of $L^\dagger L$ is bounded away from zero. (We recall that the essential spectrum includes the continuous spectrum and any eigenvalue of infinite multiplicity.) Then the kernels are finite dimensional and $\mathcal{G}$ is again bounded by $1 + \lambda_1^{-1/2}$, where λ_1 is the smallest nonzero element of the spectrum of $L^\dagger L$. It is well known that the essential spectrum is bounded away from zero whenever $L^\dagger L$ has the form $\Delta + V$, where for two positive constants c and R, V satisfies $V(x) > c$ for x outside a fixed compact set. All the operators we consider in this paper have this form.

A basic result in index theory (eg [26] p.122), is the following.

Proposition 5.2 *Let L_t, $t \in [0,1]$ be a continuous family of Fredholm operators. Then $index(L_0) = index(L_1)$.*

Thus one can sometimes deform an index problem to a more tractable index computation. To avoid potential confusion, we recall the notion of continuity assumed in the above proposition. L_t is a continuous family of operators if for each s and for every $\epsilon > 0$, there exists $\delta > 0$ so that $\|L_t F - L_s F\|/\|F\| < \epsilon$ for all nonzero F if $|t - s| < \delta$. In particular, we note that we require the δ to be independent of F. Hence, for example, the super harmonic oscillator in one variable $\psi_1 \frac{d}{dx} + \psi_2 x$, cannot be continuously deformed to $\psi_1 \frac{d}{dx} + \psi_2$ by scaling away the interaction term. If one uses the graph norm for $\psi_1 \frac{d}{dx} + \psi_2$, then the oscillator is unbounded and hence clearly cannot be deformed to a bounded operator. If one instead uses the graph norm for the oscillator, it is easy to see that all frequencies give equivalent norms and by construction, multiplication by x (as a map to E_2 is continuous in each of these norms. Hence the deformation

$$L_t = \psi_1 \frac{d}{dx} + \psi_2((1-t)x + t)$$

is continuous. The limit operator, however, is not Fredholm as a map from $E_1 \rightarrow E_2$ even though it is easy to show that it is Fredholm if the oscillator graph norm is replaced by the $\psi_1 \frac{d}{dx} + \psi_2$ graph norm.

In analyzing continuous families of operators L_t it is often useful to utilize also families of parametrices P_t. If, however, we choose P_t to be the Greens operator $\mathcal{G}_t$

of L_t then we will be plagued by the possibility that if eigenvalues converge to zero, $\mathcal{G}_t$ will become unbounded. Hence, for no other reason than to avoid such problems of bounding P_t, it is useful to define a modified Greens operator

$$\mathcal{G}_{L_t,\epsilon} = \mathcal{G}_t(I - \Pi_{L_t,\epsilon}),$$

where $\Pi_{L_t,e}$ denotes the projection onto the $\lambda \le \epsilon$ eigenspaces of $L_t^\dagger L_t$. This operator is bounded by $1 + \epsilon^{-1/2}$ and is a parametrix as long as ϵ lies below the essential spectrum.

5.3 Deforming the Index

For several of the index problems we will be considering, it seems likely that one can simply deform the given operator into a standard superharmonic oscillator and then immediately deduce the index. There are some minor issues fitting such a deformation into a continuous family. We will not treat those here because for one of our index computations - that of the noncommutative instanton - there is no single model operator to which to deform. Instead we will use the deformation invariance of the index to localize all the problems to an elementary computation around the zeros of our triholomorphic vector field G.

The case of interest to us then is D a Dirac operator anticommuting with an involution τ, L the restriction of D to the $+1$ eigenspace of τ, and E_1 and E_2 the spaces of sections of the associated bundles with finite graph and L_2 norms respectively. In this context, Fredholmness follows from the conditions in the preceding sections guaranteeing a mass gap (i.e., bounding the essential spectrum of $D^\dagger D$ away from 0.)

The deformations we will consider involve replacing G by TG for some T large. This fits into the above framework without modification since $T \ge 1$ ensures preservation of the mass gap. Moreover, scaling G is clearly continuous because the norm of G is a bounded in the given metric. Recall that G enters the Dirac operators in the form of operators,

$$\lambda^m G_m \tag{59}$$

or

$$(\sqrt{i}\,\phi^m \pm \sqrt{-i}\,\phi^{*m})G_m \tag{60}$$

which are Clifford multiplications by G. Denote these operators by $\hat{G}$. We see that the sup norm of the difference between two Dirac operators (associated to TG and SG) is bounded by $\|(T\hat{G} - S\hat{G})f\| \le |(T - S)| \times |G|_{sup} \times \|f\|$, which clearly gives

the desired inequalities for the continuity of the deformation. We note that even had the metric allowed for unbounded $|G|$, we still would have $\hat{G}$ bounded as an operator from E_1 (equipped with the graph norm) to E_2, as in the oscillator example of the previous section.

In addition, we will modify the metric on certain compact subsets. This modification may change the actual domain and range of our operator. For example τ, and hence its eigenspaces may vary with the metric. Nonetheless, we may choose quasi-isometries between them. Thus if we have Fredholm operators $D_T : E_1(T) \to E_2(T)$ and quasiisometries $h_i(T) : E_i \to E_i(T)$ then the index of $h_2(T)^{-1} D_T h_1(T)$ is T independent by the proposition and is equal to the index of D_T since the index is unchanged under composition with bounded operators with bounded inverse. We note, although we will not need it here, that the condition that h_i be quasiisometries may be relaxed to the condition that the eigenvalues of h_i and h_i^{-1} grow at most polynomially (subexponentially even) in distance from some choice of origin. This is an easy consequence of the fact that the Fredholm estimate implies exponential decay of the elements in the L_2 kernel of D_T. (See [27]). As we will use these decay properties, let us recall them in a crude form now.

Suppose we have N points $y_i, i = 1, \cdots, N$ and a Hamiltonian of the form $H = \Delta + 4T^2 V$, with $V(x) \geq 1$ if $|x - y_i| > 1$, $i = 1, \cdots, N$. Suppose also that $Hf = \lambda_0^2 f$ for some small constant λ_0, and $f \in L_2$, say with L_2 norm 1. Let

$$|x|_m := min_{1 \leq i \leq N} |x - y_i|.$$

Then $e^{(T^2 - \lambda_0^2)^{1/2} |x|_m} f \in L_2$, and the L_2 norm of $|e^{(T^2 - \lambda_0^2)^{1/2} |x|_m} f|$ restricted to the exterior of the balls of radius $R > 1$ about the y_i is finite and bounded by $4e^{(T^2 - \lambda_0^2)^{1/2} R}$. (This is not sharp. See [27] for sharper statements.) In particular, we observe that the L_2 norm of f restricted to the complement of the balls of radius $2R$ about the y_i satisfies

$$\|f_{|B_{2R}^c}\|^2 \leq 4e^{-2(T^2 - \lambda_0^2)^{1/2} R}. \tag{61}$$

Hence, f is concentrated near the zeroes of V.

Let D_T denote our Dirac operator with G replaced by $2TG$ and the metric modified to be Euclidean in a ball of radius $10R$ some $R >> 1$ about each zero of $|G|^2$. This metric modification allows us to compare D_T to a model Dirac operator which agrees with D_T near the zeros of G and has known index. Assume, as we may by replacing G initially by a suitable multiple, that $D_T^* D_T$ has the form $\Delta + 4T^2 V$, with $V(x) > 1$ in the complement of the balls of radius 1 about each zero of G. As in the previous

section, for ϵ below the continuous spectrum of D_T^2, $\Pi_{D_T,\epsilon}$ denotes the projection onto the $\lambda \leq \epsilon$ eigenspaces of D_T^2. Then

$$\text{Index} D_T^+ = \int dx \text{ tr } \tau \Pi_{T,\epsilon}(x,x). \tag{62}$$

Using (61) we see that for $d_\epsilon := \text{rank} \, \Pi_{D_T,\epsilon}$,

$$\int dx \text{ tr } \tau \Pi_{D_T,\epsilon}(x,x) = \int_{|x|_m < 2c} dx \text{ tr } \tau \Pi_{D_T,\epsilon}(x,x) + O(d_\epsilon e^{-2c(T^2-\epsilon)^{1/2}}), \tag{63}$$

for some choice of c. Hence it suffices to bound d_ϵ independently of T large and to compute the integral of $\text{tr } \tau \Pi_{T,\epsilon}(x,x)$ over $|x|_m < 2c$ in the large T limit.

First we estimate d_ϵ. Let $D_T^2 f = \lambda^2 f$, for $\lambda \leq \epsilon$, and $\|f\|_{L_2} = 1$. Let Q_T denote the Green's operator for the super harmonic oscillator (SHO) which agrees with D_T^2 in a neighborhood of radius $4R$ about the zeros of G. Let ρ_R denote a cutoff function supported on a ball of radius $2R$ where D_T^2 is the SHO and identically one on a ball of radius R. Then $(\rho_R f - Q_T D_T^2 \rho_R f)$ is in the kernel of the SHO. Denote its norm by a and introduce a unit vector v in the kernel of SHO such that

$$(\rho_R f - Q_T D_T^2 \rho_R f) = av.$$

Observe that Q_T has sup norm $\leq T^{-1}$. Now consider the equality

$$D_T^2 \rho_R f = \lambda^2 \rho_R f + [\Delta, \rho_R] f.$$

By our assumptions, the right hand side is $O(\lambda^2) + O(e^{-TR})$ (not sharp). Hence $\|\rho_R f - av\| = O(\lambda^2)$ for $\lambda > O(e^{-TR/2})$. Setting, for example, $\epsilon = 1/T$, we have then

$$\|f - av\| = O(1/T^2). \tag{64}$$

Moreover, such an inequality is true for any vector in the image of $\Pi_{D_T,1/T}$. We conclude then that rank $\Pi_{D_T,1/T}$ is no larger than the dimension of the kernel of the SHO (times the number of zeros of G). This bounds $d_{1/T}$ and completes our demonstration that it suffices to compute the trace over a bounded region.

In the following, for simplicity of notation we will consider the case of a single zero for G, but the general case follows similarly with only notational complications.

Let S_T^+ and τ_E denote the Euclidean Dirac operator and involution which agree with D_T^+ and τ near the zeros of G. Let F_T denote the Greens operators for S_T^+. To define a parametrix for D_T, introduce $\mathcal{G}_T$, the Greens operator for D_T^+, and let P_T be the modified Greens operator

$$P_T := \mathcal{G}_T(I - \Pi_{D_T,1/T}).$$

Define

$$I_1 := \text{Index}(D_T^+) - \text{Index}(S_T^+) = Tr([D_T^+ P_T - S_T^+ F_T] - [P_T D_T^+ - F_T S_T^+]).$$

Then we wish to show that the integer $I_1 = 0$. By (63), the above traces can be approximated for R, T large as

$$I_1 = \text{Tr}\rho_R([D_T^+ P_T - S_T^+ F_T] - [P_T D_T^+ - F_T S_T^+]) + O(d_{1/T} e^{-RT/2}).$$

On the support of ρ_R, $D_T^+ = S_T^+$, hence we have

$$I_1 = \text{Tr}\rho_R[D_T^+, P_T - F_T] + O(d_{1/T} e^{-RT/2}).$$

Using the cyclic property of the trace, we rewrite the first term on the right hand side of the above formula as

$$\text{Tr}\rho_R[D_T^+, P_T - F_T] = -Tr[D^+, \rho_R](P_T - F_T) + \int dx \, \nabla_i V(x)^i.$$

where $V(x)$ is the vector with

$$V_i(x) := \text{Tr}\gamma_i(\rho_R(x)(P_T - F_T)(x,x)).$$

The integral vanishes by Stoke's theorem, leaving

$$I_1 = -\int \text{tr}[D^+, \rho_R](x)(P_T - F_T)(x,x)dx.$$

We estimate the last term by converting it back into an expression involving the exponentially decaying projection operators. Write

$$\begin{aligned}
& [D_T^+, \rho_R](P_T - F_T) \\
= \; & [D_T^+, \rho_R](F_T S_T^+ + \Pi_{S_T^+, 1/T})(P_T - F_T) \\
= \; & [D_T^+, \rho_R]F_T(S_T^+ P_T - S_T^+ F_T) + [D_T^+, \rho_R](\Pi_{S_T^+, 1/T})(P_T - F_T). \qquad (65)
\end{aligned}$$

The last term is $O(Te^{-RT})$ because

$$[D_T^+, \rho_R](\Pi_{S_T^+, 1/T}) = O(e^{-RT})$$

by (61) and because $(P_T - F_T)$ has sup norm $< T$ by construction. We separate the first term into two additional terms

$$\begin{aligned}
& [D_T^+, \rho_R]F_T(S_T^+ P_T - S_T^+ F_T) \\
= \; & [D_T^+, \rho_R]F_T(D_T^+ P_T - S_T^+ F_T) + [D^+, \rho_R]F_T(S_T^+ - D_T^+)P_T. \qquad (66)
\end{aligned}$$

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The first term is again exponentially decreasing because $(D_T^+ P_T - S_T^+ F_T)$ is a difference of exponentially decaying projection operators and F_T is uniformly bounded. We can compute F_T explicitly and it is $O(e^{-T\delta})$, where δ is the distance between the support of $[D_T^+, \rho_R]$ and the support of $(S_T^+ - D_T^+)$. Hence all the terms are exponentially decreasing, and we deduce that I_1 is exponentially decreasing. On the other hand, it is the difference between two integers and must therefore vanish.

We summarize our results.

$$\text{Index}(D_T^+) = \text{Index}(S_T^+).$$

When there is more than one zero of G, a minor variation of the same argument yields

$$\text{Index}(D_T^+) = \sum_i \text{Index}(S_T(i)^+),$$

where $S_T(i)$ is the local model for D_T at the ith zero of G. In order to extend our argument to this case, we must replace expressions of the form $F_T P_T$ and $\Pi_{S_T^+, 1/T} P_T$ in the previous expression by $F_T \rho_{nT} P_T$ and $\Pi_{S_T^+, 1/T} \rho_{nT} P_T$ for some large n, because F_T and $\Pi_{S_T^+, 1/T}$ need not extend naturally to the full moduli space in the many zero case. This will introduce new error terms of the form $\nabla \rho_{nT} P_T \rho_T to be estimated$. Our decay estimates can once again be used to show these terms are also exponentially decaying.

5.4 Computing the Deformed Index

We now use the deformation arguments of the preceding section to complete the index computations. We consider first the case of the quantum mechanics with 4 complex supersymmetries on a moduli space of dimension 4k and compute $\mathcal{I}_s^+$. The other cases are very similar and follow with minor modifications. In the $\mathcal{I}_s^+$ case, we have reduced the problem to computing the index of the operator $S_{1/e} := d - \iota_G + i(d^\dagger - \iota_G^\dagger)$ acting on selfdual forms ($\tau_+ = 1$) on C^{2k}. Separating variables, we see that the index of S_1 is the product of the indices of the D_c, $c = 1, \cdots, k$, where where $D_c := d - \iota_{a_c K_c} + i(d^\dagger - \iota_{a_c K_c}^\dagger)$ (no sum over c) acting on selfdual forms on C^2. Using the deformation invariance of the index again, we may assume $a_c = 2$.

This latter index is easy to calculate exactly as follows. We compute

$$\begin{aligned} & D_c D_c^\dagger + D_c^\dagger D_c \\ & = \Delta + |2K_c|^2 - \{d, \iota_{2K_c}^\dagger\} + i\{\iota_{2K_c}, d\} - i\{d^\dagger, \iota_{2K_c}^\dagger\} - \{d^\dagger, \iota_{2K_c}\}. \end{aligned} \quad (67)$$

Let z_1 and z_2 denote complex coordinates on C^2. Then $2K_c = i\sum_{j=1}^{2}(z_c\frac{\partial}{\partial z_c} - \bar{z}_c\frac{\partial}{\partial \bar{z}_c})$. Hence, $|2K_c|^2 = |z_1|^2 + |z_2|^2$. In the coordinate frame, on (p,q) forms we have

$$i\{\iota_{2K_c}, d\} = -i\{d^\dagger, \iota_{2K_c}^\dagger\} = i\mathcal{L}_{2K_c} = -(p-q) - 2K_c/i,$$

and

$$\{d, \iota_{2K_c}^\dagger\} = idz_1 d\bar{z}_1 + idz_2 d\bar{z}_2,$$
$$\{d^\dagger, \iota_{2K_c}\} = idz_1^\dagger d\bar{z}_1^\dagger + idz_2^\dagger d\bar{z}_2^\dagger. \tag{68}$$

Hence, we have

$$D_c D_c^\dagger + D_c^\dagger D_c$$
$$= \Delta + |z|^2 - 2(p-q) - 4K_c/i - \{d, \iota_{2K_c}^\dagger\} - \{d^\dagger, \iota_{2K_c}\}. \tag{69}$$

The functions

$$f(a,b,c,d) := (\partial_{z_1} - \bar{z}_1)^a(\partial_{\bar{z}_1} - z_1)^b(\partial_{z_2} - \bar{z}_2)^c(\partial_{\bar{z}_2} - z_2)^d e^{-|z|^2/2}, \tag{70}$$

with $a,b,c,d \geq 0$, span the eigenspace of $\Delta + |z|^2$ with the eigenvalue $2(a+b+c+d)+4$. We compute commutators to obtain

$$2K_c f(a,b,c,d) = i(-a+b-c+d)f(a,b,c,d).$$

Therefore, on the algebraic span of $f(a,b,c,d)$ we have

$$D_c D_c^\dagger + D_c^\dagger D_c = 4(a+c) + 4 - 2(p-q) - \{d, \iota_{2K_c}^\dagger\} - \{d^\dagger, \iota_{2K_c}\}. \tag{71}$$

This vanishes if and only if $a = c = 0$ and the differential form coefficient of $f(a,b,c,d)$ takes one of the following forms:

$$dz_1 \wedge dz_2, \tag{72}$$

or

$$1 + (idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2)/2 - dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$

or

$$(dz_1 + idz_2 \wedge d\bar{z}_2 \wedge dz_1/2),$$

or

$$(dz_2 + idz_1 \wedge d\bar{z}_1 \wedge dz_2/2).$$

We have, therefore, an infinite dimensional kernel to our operator before taking into account the constraint on charge. We now recall that we wish to restrict to the space

$$2q_c = \mathcal{L}_{2K_c}/i = (p - q) + 2K_c/i.$$

With the above normalization of K_c, q_c's are integers or half-integers. On $f(0, b, 0, d)$ this imposes the constraint

$$2q_c = (p - q) + (b + d).$$

The index of D_c is thus given by the number of ways to choose nonnegative integers b and d so that

$$b + d = 2q_c - (p - q)$$

with $p - q = 2$ or 0 plus twice the number of ways to choose nonnegative integers b and d so that

$$b + d = 2q_c - 1.$$

There are $8q_c$ such solutions for positive q_c, one such solution for $q_c = 0$, and none for negative q_c. All of these solutions are self-dual, so the index of D_c is

$$\text{Index}(D_c) = \begin{cases} 8q_c & q_c > 0 \\ 1 & q_c = 0 \\ 0 & q_c < 0 \end{cases}.$$

Note that this result assumes a positive coefficient of K_c. For a negative coefficient, the computation proceeds exactly as above, provided that we make the following exchanges of coordinates,

$$\begin{aligned} z_1 &\leftrightarrow \bar{z}_1 \\ z_2 &\leftrightarrow \bar{z}_2 \end{aligned} \tag{73}$$

This maps K_c to $-K_c$, and flips the sign of q_c in the charge constraint above. In other words, the sign condition in the index formula is really on $a_c q_c$ for each j. Thus the index is

$$\text{Index}(D_c) = \begin{cases} 8|q_c| & a_c q_c > 0 \\ 1 & a_c q_c = 0 \\ 0 & a_c q_c < 0 \end{cases}.$$

for each c.

Thus, whenever there exist a massgap, the index $\mathcal{I}_s^+$ is

$$\mathcal{I}_s^+ = \sum \left(\prod_c \begin{cases} 8|q_c| & a_c q_c > 0 \\ 1 & a_c q_c = 0 \\ 0 & a_c q_c < 0 \end{cases} \right).$$

where the sum is over the zeros of the potential. Note that the index is nonvanishing only if all $a_c q_c$ (no summation) are nonnegative. The states in the kernel of the Dirac operator must be annihilated by $\mathcal{H} - \mathcal{Z}$ as well, and the central charge $\mathcal{Z}$ of the states

$$e \sum_c a_c q_c > 0 \tag{74}$$

equals the energy.

Computation of $\mathcal{I}_s^-$, appropriate for those states with positive central charge, proceeds similarly. In fact, this problem can be mapped to that of $\mathcal{I}_s^+$ by

$$
\begin{aligned}
\varphi &\rightarrow \varphi^* \\
\varphi^* &\rightarrow \varphi \\
K_c &\rightarrow -K_c
\end{aligned}
\tag{75}
$$

The net effect is to flip the sign condition on the charges q_c, so

$$\mathcal{I}_s^- = \sum \left(\prod_c \left\{ \begin{array}{ll} 8|q_c| & a_c q_c < 0 \\ 1 & a_c q_c = 0 \\ 0 & a_c q_c > 0 \end{array} \right\} \right).$$

whenever a massgap exists. The sum is over zeros of the potential. The energy of the contributing states is $-e \sum_c a_c q_c > 0$.

We consider next the same set of operators but now restricted to the $+1$ eigenspace of τ_4. We see that the terms with $p + q$ even are in the $+1$ eigenspace of τ_4, and the terms with $p + q$ odd are in the -1 eigenspace of τ_4. This leads to zero index for all nonzero q_c. When $q_c = 0$, we get a solution with $p - q = 0$. There is only one of these. The index of τ_4 is then

$$\mathcal{I}_4 = \sum \left(\prod_c \left\{ \begin{array}{ll} 1 & q_c = 0 \\ 0 & q_c \neq 0 \end{array} \right\} \right)$$

where the sum is over the zeros of the potential.

Finally, we consider the minor modifications necessary to compute $\mathcal{I}_2$. Once again a separation of variables allows us to reduce the index of the Euclidean operator to a product of indices of operators B_c, $l = 1, \cdots, k$ on C^2. In coordinates, B_c has the form

$$B_c = \sum_{j=1}^{2} [\lambda_{2j-1}(\frac{\partial}{\partial x_j} + iy_j) + \lambda_{2j}(\frac{\partial}{\partial y_j} - ix_j)],$$

acting on the $+1$ eigenspace of $4\lambda_1 \lambda_2 \lambda_3 \lambda_4$. This choice of the Dirac operator corresponds to positive coefficients $a_c = 2$ for all $c = 1, \ldots, k$. Then in a covariant constant

frame,

$$2B_cB_c^\dagger + 2B_c^\dagger B_c = \Delta + |z|^2 + 4iK - 4i\sum_j \lambda_{2j-1}\lambda_{2j}.$$

Acting on the algebraic span of $f(a,b,c,d)$,

$$2B_cB_c^\dagger + 2B_c^\dagger B_c = 4(a+c) + 4 + 4i\sum_j \lambda_{2j-1}\lambda_{2j}.$$

This has infinite dimensional kernel spanned by the product of $f(0,b,0,d)$ and a co-variant constant spinor in the (one dimensional) intersection of the $-1/2$ eigenspaces of $i\lambda_1\lambda_2$ and $i\lambda_3\lambda_4$.

The charge constraint in a covariant constant frame takes the form

$$2q_c = 2K_c/i - i\lambda_1\lambda_2 - i\lambda_3\lambda_4.$$

Acting on the above basis elements of the kernel of $2B_cB_c^\dagger + 2B_c^\dagger B_c$ this reduces to

$$2q_c = b + d + 1.$$

Counting as before this yields a $2q_c$ dimensional kernel which lies entirely in the $+1$ eigenspace of $4\lambda_1\lambda_2\lambda_3\lambda_4$. Hence, the index of B_c is

$$\text{Index}(B_c) = \left\{ \begin{array}{ll} 2|q_c| & a_cq_c > 0 \\ 0 & a_cq_c \le 0 \end{array} \right\}.$$

Thus, whenever there exist a massgap, the index $\mathcal{I}_2$ is

$$\mathcal{I}_2 = \sum \left(\prod_c \left\{ \begin{array}{ll} 2|q_c| & a_cq_c > 0 \\ 0 & a_cq_c \le 0 \end{array} \right\} \right).$$

Finally, we conclude the index computations by noting that they reproduce the four-dimensional results summarized in the previous section. In fact, the wavefunctions found in Ref. [25] and in Ref. [24] can be seen easily to reduce to the superharmonic oscillator wavefunctions above in the limit of $\tilde{a} \gg q$.

6 BPS Bound States

The above index computations count differences in the number of ground states with respect to Z_2 involutions $\tau_\pm, \tau_4, \tau_2$,

$$\text{Index} = n_+ - n_- \tag{76}$$

where $n_\pm$ are the number of ground states with τ eigenvalue ± 1. We are actually interested in the sum $n_+ + n_-$ instead, for which one needs a more refined understanding of the dynamics. For the case of τ_2 and $\tau_\pm$, we anticipate n_- vanishes by itself. Such a vanishing theorem is shown rigorously for the simplest cases in Appendix. We will assume in this section that $n_- = 0$ holds true for τ_2 and $\tau_\pm$ in all cases, and compare the results to what are expected on physical grounds.

6.1 $N = 4$ Yang-Mills Theories

The supersymmetric quantum mechanics with four complex supercharges describe dynamics of monopoles in $N = 4$ Yang-Mills theories. Recent studies of D-branes indicates the following three possibilities for dyonic bound states of monopoles.

- The state is 1/2 BPS in the Yang-Mills field theory. These states would be annihilated by all supercharges of the low energy monopole dynamics, which is possible only if the central charges in the relative part of the dynamics is absent. This is guaranteed when all relative electric charge q_a's vanish. In particular, this includes purely magnetic bound states.

- The state is 1/4 BPS in the Yang-Mills field theory. These states would be annihilated by half of the supercharges of the low energy monopole dynamics and not by the other half. This is possible only if at least one central charge is nonzero.

- The state is non-BPS.

The index computation of the previous section tells us something about 1/2 BPS and 1/4 BPS states, where we counted indices $\mathcal{I}_4$ and $\mathcal{I}_s^\pm$ in the special limit where only one G^I, say G^5, is turned on. Equivalently, we considered vacua where two Higgs fields are turned on.

Of the three indices, only $\mathcal{I}_4$ is robust against turning on more than one G^I's. The Dirac operator $iQ \pm Q^\dagger$ would no longer anticommute with $\tau_\pm$ but does anticommutes with τ_4. Only $\mathcal{I}_4$ is a well-defined index in such generic vacua. Turning on additional G^I always increases the massgap, and is a Fredholm deformation that preserves $\mathcal{I}_4$. Thus our result shows that, in generic vacua,

$$\mathcal{I}_4 = 1, \tag{77}$$

when $q_a \equiv 0$, and zero otherwise. Since the central charge of the state that contributes to the index is zero, the state must be annihilated by all supercharges of the quantum mechanics and is a 1/2 BPS in $N = 4$ Yang-Mills theory.[3] This is consistent with the existence of a unique magnetic 1/2 BPS bound state of monopoles in generic Coulomb vacua, which is expected from the $SL(2, Z)$ electromagnetic duality. One of the generators of $SL(2, Z)$ maps massive charged vector multiplets to purely magnetic bound states in 1-1 fashion. After taking into account the automatic degeneracy 16 from the free center-of-mass fermions, the total degeneracy of these bound states is alway 16, which fits the $N = 4$ vector multiplet nicely. This purely magnetic bound state was previously constructed by Gibbons [5] in special vacua where all G^I's vanishes.

Existence of 1/4 BPS states are more sensitive to the vacuum choice and the electric charges. The existence criteria were first found by Bergman [11], where he constructed these dyons as string webs ending on D3-branes. The first necessary condition is that the string web should be planar, which is equivalent to the condition that, effectively, only one linearly independent G^I is present. This allows us to assume without loss of generality that only G^5 is turned on, as far as counting 1/4 BPS states are concerned. Thus, the computation of $\mathcal{I}_s^\pm$ in the previous section is directly applicable.

Secondly, at each junction of the string web, the string tensions must balance against each other, which in the present language of low energy dynamics translates to the condition that the effective potential in the charge-eigensector is nonrepulsive along all asymptotic directions [12];

$$|q_c| \leq |\tilde{a}_c|. \tag{78}$$

This second condition may indicate the existence of a minimal energy bound state, however, does not guarantee that the state would preserve some supersymmetry.

Finally, a minimal energy configuration is supersymmetric when the orientation of string segments are consistent with each other. Say, if one fundamental string segment is directed to one particular direction, then another fundamental string in the same web must be directed the same way. The second string can point toward

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[3]One might think that existence of this bound state is obvious since the potentials are all attractive and also there exists a classical BPS monopole of the same magnetic charge. However, none of these guarantee the existence of BPS bound state at quantum level. In fact, the same set of facts are true for a pair of distinct monopoles in $N = 2$ $SU(3)$ Yang-Mills theory but we know that such a purely magnetic bound state does not exist as a BPS state [8].

the opposite direction and still balance the tension, but such a combination breaks all supersymmetry. This orientation condition on the string web, is nothing but the condition that the product $a_c q_c$'s (no summation) are all of same sign. See figure 1. Thus, a 1/4 BPS dyon may exist only when $|q_c| \leq |\tilde{a}_c|$ for all c and $a_c q_c$ are all of same sign, at least one of which is nonzero.

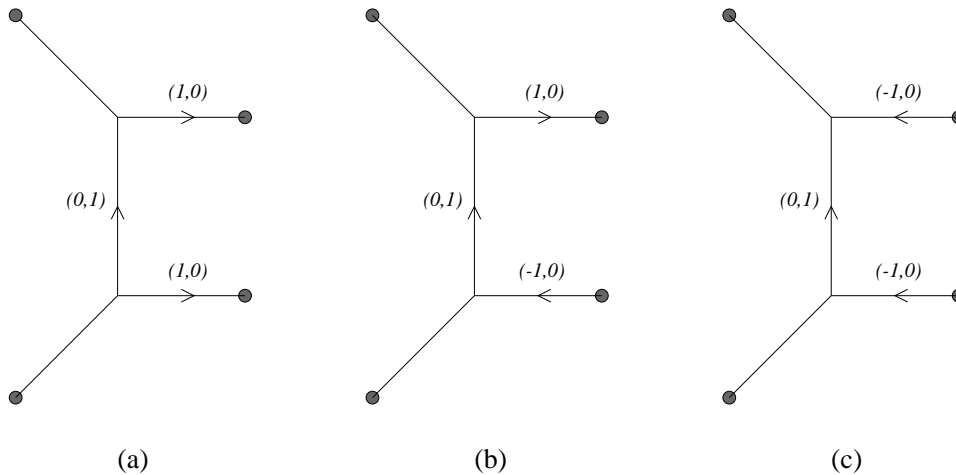(a)   (b)   (c)

Figure 1: Simple webs of (q, p) strings that represent dyons in $N = 4$ $SU(4)$ theory. The filled circles represent D3-branes, while vertical lines are D-strings, and horizontal lines are fundamental strings. Configurations (a) and (c) preserve a quarter of supersymmetry that was left unbroken by D3's, while (b) breaks all supersymmetry. When translated to Yang-Mills field theory on D3's, the horizontal separations between D3-branes are encoded in a_c's while the electromagnetic charges are determined by which string ends on which D3.

The indices $\mathcal{I}_s^{\pm}$ were computed with the massgap condition $|q_c| < |\tilde{a}_c|$ to begin with, and yielded nonzero value only when all $a_c q_c$ were of the same sign; For positive signs of $a_c q_c$, $\mathcal{I}_s^+ \neq 0$, while for negative $a_c q_c$'s, we have $\mathcal{I}_s^- \neq 0$. The result is clearly consistent with the existence criteria set by the string-web construction, and furthermore gives us extra information beyond the string web picture. The index indicates that the degeneracy of such a 1/4 BPS state is

$$16 \times \prod_c \mathrm{Max}\,\{8|q_c|, 1\}. \tag{79}$$

The factor 16 arises from the free center-of-mass fermions.

In the two monopole bound states, the number $8|q|$ is accounted for by four angular momentum multiplets of $j = |q|, |q| - 1/2, |q| - 1/2, |q| - 1$ [25].[4] The top angular

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[4]The first three suffices for $|q| = 1/2$.

momentum $|q|$ in the relative part of the wavefunction has a well-known classical origin: when an electrically charged particle moves around a magnetic object, the conserved angular momentum is shifted by a factor of $eg/4\pi$. While fermions can and do contribute, the number of fermions scales with the number of monopoles, and not with the charge q_a. In fact, it is most likely that the top angular momentum of such a dyonic bound state wavefunction is

$$j_{top} = \sum_c |q_c|, \tag{80}$$

for large charges, so that the highest spin of the dyon would be

$$1 + j_{top} = 1 + \sum_c |q_c|, \tag{81}$$

after taking into account the universal vector multiplet structure from the free center-of-mass part. On the other hand, a 1/4 BPS supermultiplet with the highest spin $j_{top} + 1$ has the total degeneracy of [12]

$$16 \times 8 \sum_c |q_c|, \tag{82}$$

which is much less than the number of states we found above unless all but one q_c vanishes. Thus, this implies that there are many 1/4 BPS, thus degenerate, super-multiplets of dyons for a given set of electromagnetic charges. This is probably the least understood of our results. While one would expect to find degenerate states within a supermultiplet, there is no natural symmetry that accounts for the existence of many supermultiplets of the same electromagnetic charges and of the same energy.

For large electric charges q_a, the number of dyon supermultiplets scales as, at least,

$$\left(\prod_c \text{Max}\left\{8|q_c|, 1\right\}\right) / \left(8\sum_c |q_c|\right). \tag{83}$$

Proliferation of dyonic states of a given charge was anticipated by Kol some time ago in the context of string webs in 5 dimensions [28]. Because our computation was performed for a collection of distinct monopoles, which put some constraint on the possible topology of the related string web, it is not immediately clear to us whether we can make any sensible statement in the regime where Kol's prediction is applicable.[5] Nevertheless, it is tantalizing that we found the number of states

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[5]Kol anticipated exponentially large numbers of states, in fact, which is much more than our powerlike result.

increasing much faster than would have been expected from supersymmetry alone. It is not clear to us why this happens and what interpretation this may have in the Yang-Mills field theory.

In the regime where $|q_c| \geq |\tilde{a}_c|$ for some q_c, we cannot rely on the current index computation. On the other hand, since even a single repulsive direction, i.e., $|q_c| > |\tilde{a}_c|$ for some c, prohibits a bound state (supersymmetric or not), the unresolved question boils down to the marginal case, where $|q_c|$ equals $|\tilde{a}_c|$ for some c's while the others satisfy $|q_c| < |\tilde{a}_c|$. The only state that must exist for sure is the purely magnetic bound state ($q_a = 0$), which was constructed by Gibbons when $a_c \equiv 0$ and which is necessary for $SL(2, Z)$ invariance. The explicit construction of two-monopole bound states in Ref. [25] seem to indicate that no *dyonic* bound state may form along such marginal directions, but this remains to be seen for multi-monopole cases.

6.2 $N = 2$ **Yang-Mills Theories**

In $N = 2$ Yang-Mills theories, a state could be either BPS or non-BPS. There is no such thing as a 1/4 BPS state. Dyons that would have been 1/4 BPS when embedded in $N = 4$ theories, are realized as either 1/2 BPS or non-BPS depending on the sign of the electric charges. The index of this Dirac operator was nonzero only when

$$0 < q_c < \tilde{a}_c \quad \text{for all } c$$
$$\text{or} \quad 0 > q_c > \tilde{a}_c \quad \text{for all } c \tag{84}$$

which gives us a possible criterion for BPS dyon to exist. This condition is similar to the condition for BPS dyons or monopoles to exist in $N = 4$ Yang-Mills theories but differs in two aspects. The first is that given a set of a_c's, all of which are positive (negative), the electric charge q_c's must be all positive (negative).

The second difference from $N = 4$ case is that a purely magnetic bound state of monopoles does not seem to exist as a BPS state, even though there exists a classical BPS solution of such a charge. This feature was noted previously in Ref. [8], where bound states of a pair of distinct monopoles were counted explicitly. In fact, the index indicates that all relative q_a must be nonvanishing for a BPS state to exist. Assuming the vanishing theorem, the number of BPS dyonic bound state under the above condition is

$$4 \times \prod_c 2|q_c|, \tag{85}$$

The overall factor 4 is from the quantization of the free center-of-mass fermions.

For large electric charges we again observe the proliferation of supermultiplets. The top angular momentum and thus the size of the largest supermultiplet can grow only linearly with $\sum |q_c|$ which means that the number of supermultiplets with the same electric charges scales at least as

$$\left(\prod_c 2|q_c| \right) / \left(2 \sum_c |q_c| \right) \tag{86}$$

for large q_c's. Again it is not clear to us what implication this has in the Yang-Mills field theory.

In the regime where $|q_c| \geq |\tilde{a}_c|$ for some q_c, again we cannot rely on the current index computation. For the same reason as in $N = 4$ Yang-Mills theory, no bound state can exist if even a single repulsive direction ($|q_c| > |\tilde{a}_c|$ for some c) exists, so the unresolved question boils down again to the marginal case, where $|q_c|$ equals $|\tilde{a}_c|$ for some c's while the others satisfy $|q_c| < |\tilde{a}_c|$. Extrapolating from the explicit construction of two-monopole bound states in Ref. [8], we suspect that no bound state may form along such marginal directions.

6.3   Ground States of a Noncommutative Instanton Soliton

Supersymmetric ground states and excited BPS states of an instanton soliton in $S^1 \times R^{3+1}$ can be counted similarly as above. The only difference as far as computing the index goes, is that the potential has many zeros. For a single instanton in noncommutative $U(n)$ theory, there are precisely n zeros of the bosonic potential, and near each of these zeros, the Dirac operator can be deformed to that of a superharmonic oscillator. One crucial difference in interpreting the result in physical terms comes from identification of the conserved charges. Of n possible conserved $U(1)$ charges, $n - 1$ relative charges are again electric charges. However, the overall conserved $U(1)$ does not correspond to a gauge symmetry, and comes from translation of the instanton along S^1. This last $U(1)$ charge is just the Kaluza-Klein momentum along S^1.

Of particular interest are the quantum ground state of the instanton with no $U(1)$ charges excited. In the maximally supersymmetric $U(n)$ Yang-Mills theory, the index tells us that there are n distinct BPS supermultiplets of ground states. This result was anticipated in Ref. [19]. With half as much supersymmetry, however, the index is consistent with no supersymmetric quantum ground state exist at all.

Quantum states of instanton soliton in R^4 was previously studied in the commutative setting [29, 30]. In particular, the absence of a quantum ground state of

instanton soliton in the nonmaximal supersymmetric Yang-Mills theories, has been observed from string-web construction.

7   Conclusion

By computing indices and assuming vanishing theorems, we counted supersymmetric bound states of arbitrarily many distinct monopoles in $N = 2$ pure Yang-Mills theories and also in $N = 4$ Yang-Mills theories. The relevant low energy dynamics are supersymmetric sigma-models with potential(s), where the supercharges preserved by supersymmetric bound states can be interpreted as Dirac operators twisted by triholomorphic Killing vector fields. An obvious generalization of this computation is to include hypermultiplets in $N = 2$ Yang-Mills theories, but it goes beyond the scope of this paper.

Counting of 1/2 BPS states in $N = 4$ Yang-Mills yielded a result consistent with electromagnetic duality of the theory. In particular, the necessary purely magnetic bound states of distinct monopoles are all accounted for in $SU(n)$ theories. While this result is not surprising, it is still significant in that this was shown for the first time in all generic Coulomb vacua of the Yang-Mills theory. In contrast, distinct $N = 2$ monopoles do not seem to bind at all unless all possible relative charges are turned on.

Existence criteria for 1/4 BPS states, previously found in the context of string-webs, are also faithfully reflected in the index formulae. On the other hand, the degeneracy of most 1/4 BPS dyons is shown to be much larger that one would have expected from a single 1/4 BPS supermultiplet with a physically reasonable angular momentum. $N = 2$ Dyons of the same electromagnetic charges as 1/4 BPS dyons of $N = 4$ theories, could be BPS or non-BPS, depending on the signs of the electric charge. We also counted the degeneracy of such $N = 2$ BPS dyons, which shows similar proliferation of supermultiplets. This phenomenon is not understood at the moment. It should be also interesting to see how the degeneracy behaves when both electric and magnetic charges are large.

Appendix

Here, we prove a vanishing theorem for τ_2 and $\tau_\pm$ on four-dimensional moduli space. Let us consider τ_2 first. Because of the triholomorphic Killing conditions on G, dG is self-dual and does not couple to antichiral spinors. Then the Dirac operator is a simple Laplacian;

$$- D_m D^m \tag{87}$$

when acting on antichiral spinors. Using the standard trick of sandwiching this operator by a hypothetical zero mode Ψ and its complex conjugate Ψ^*, we find

$$0 = -\int \prod_m dz^m \; \Psi^* \nabla_m D^m \Psi = -\int \prod_m dz^m \; g^{mn}(D_n \Psi)^*(D_m \Psi) \tag{88}$$

where the possible boundary term vanishes by itself since the massgap forces Ψ to be exponentially small at large distances. Therefore,

$$0 = D_m \Psi = (\nabla_m - i G_m)\Psi \tag{89}$$

everywhere. This modified connection is still unitary as G_m is real. Hence, Ψ is covariant constant with respect to metric compatible connection and is therefore of constant norm. Such an f cannot be normalizable on an infinite volume space unless it is identically zero, which proves the vanishing theorem in four dimensions for τ_2.

The case of $\tau_\pm$ can be handled similarly. Let us recall that differential forms can be thought of as a tensor product of two spinors. With an appropriate sign convention, we can identify various sectors of the former with those of the latter as follows

- selfdual even form $\rightarrow$ [chiral spinor]⊗[chiral spinor]

- selfdual odd form $\rightarrow$ [chiral spinor]⊗[antichiral spinor]

- antiselfdual odd form $\rightarrow$ [antichiral spinor]⊗[chiral spinor]

- antiselfdual even form $\rightarrow$ [antichiral spinor]⊗[antichiral spinor]

On antiselfdual even form, then, the Clifford action of dG is trivial. Since the self-dual curvature does not couple to antiselfdual forms either, its action is also trivial. Thus, the square of the Dirac operator becomes a simple Laplacian again,

$$D_\pm^2 = -D_m D^m \tag{90}$$

with $D_m = \nabla_m \mp i G_m^5$. By the same logic as in the spinor case, therefore, no antiself-dual even-form solution can exist. Finally this also shows that antiselfdual odd-form

solution does not exist; Unless the central charge vanishes, a solution generates other solutions via the action of broken supercharges. The broken supercharges are the linear combination of Q and $Q^\dagger$ orthogonal to $D_\pm$, so that it flips τ_4 while preserving $\tau_\pm$. Thus, the number of odd-form solutions equals the number of even-form solutions, in each $\tau_\pm$ eigensectors, whenever the central charge is nonzero. This proves the vanishing theorem for $\tau_\pm$ for sectors with nonzero $U(1)$ charges q_c.

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